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An *hp*-adaptive strategy based on continuous Sobolev embeddings☆

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ABSTRACT

A new *hp*-adaptive approach for finite element methods is presented. Specifically, the paper addresses the well-known question of how to effectively decide between *h*- and *p*-refinement within an adaptive *hp*-finite element procedure. The main idea is based on testing the smoothness of the underlying PDE solution by means of suitable continuous Sobolev embeddings. Numerical experiments confirm the effectiveness of the proposed method.

1. Introduction

The focus of this work is the *hp-adaptive* approximation of solutions of differential equations by *hp*-finite element methods (FEM). Numerical solutions obtained from finite element discretizations are typically piecewise polynomials on a given partition $\mathcal{T}$ (mesh) of the underlying domain Ω of the differential equation under consideration. Generally, in order to obtain piecewise polynomial approximations of functions various approaches exist. Let us recall three well-known ideas.

- *h*-approximation: the approximation of a function by piecewise polynomials is improved by repeatedly subdividing the mesh $\mathcal{T}$ into smaller subdomains (elements), i.e., by decreasing the local size h of the partition; here, the polynomial degree on each element is kept fixed at a usually low number $p = 1$ or $p = 2$.
- *p*-approximation (or spectral approximation): in this context, approximations are improved by increasing the local polynomial degrees p on each element on a fixed mesh.
- *hp*-approximation: here, local refinement of the mesh and adjustment of the elementwise polynomial degrees is suitably combined.

Standard results (see, e.g., [1]) state that the quality of approximation of a function by, e.g., polynomials, is closely related to its (local) regularity. Hence, in the context of piecewise approximation of functions with non-uniform smoothness properties, an appropriate adjustment of the mesh and/or the local polynomial degrees to the given situation is often quite beneficial. For example, in the approximation of smooth solutions of elliptic or parabolic partial differential equations with local singularities, suitable mesh/polynomial degree combinations, i.e., *hp*-approximations, have proved to be remarkably effective. Indeed, even in the presence of certain local singularities they are able to provide high algebraic or even exponential rates of convergence; see, e.g., [2] and the reference therein. Efficient *hp*-refinement approaches are typically based on the following strategy:

(p) in regions where the function to be approximated is smooth high polynomial degrees on comparatively large elements are applied (spectral or *p*-approximation);
(h) in areas of low regularity the mesh is refined locally and low polynomial degrees are used (*h*-approximation).

In order to automate this *hp*-approximation process, i.e., to appropriately decide between *h*- and *p*-refinement, several *hp-adaptive strategies* have been suggested in the literature. The key challenge in the design of such decision schemes is

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to balance h- and p-approximation as effectively as possible and thereby to obtain high quality numerical results. Here, one possible strategy is to estimate the local smoothness of the function (e.g., the solution of a differential equation) to be approximated and then to perform h- or p-refinement in accordance with the strategies (p) and (h) above. In this context we mention the paper [3]; cf. also [4]. Indeed, the idea of using smoothness testing indicators will also be pursued in the present work. Incidentally, the first automatic *hp*-strategy for elliptic problems was presented in [5]. Further approaches can be found, for example, in [6–12].

The goal of the present paper is to propose a new *smoothness estimation technique* for *hp*-adaptive FEM. Here, the basic idea is to use suitable *continuous Sobolev embedding bounds* and to monitor the corresponding continuity constants as the *hp*-approximation space is enriched. In this way, we obtain a simple indicator in order to determine whether or not a function could be locally smooth. Accordingly, appropriate *hp*-refinements following the strategies (p) and (h) may be performed.

The article is organized as follows: In Section 2 we recall some standard notation and present a Sobolev embedding result. Moreover, in Section 3 we shall briefly revisit standard *hp*-FEM and some a posteriori error estimates for a simple elliptic model problem, and formulate the new *hp*-adaptive strategy. Section 4 presents a number of numerical test examples. Finally, we provide some conclusions in Section 5.

2. Sobolev spaces and a continuous embedding

For any open domain D we denote by $L^2(D)$ the space of all square-integrable functions on D with norm

$$\|v\|_{0,D} = \left(\int_D v(x)^2 \,\mathrm{d}x\right)^{1/2}.$$

Furthermore, for $m \in \mathbb{N}$, the Sobolev space $H^m(D)$ consists of all functions for which the norm

$$\|v\|_{m,D} = \left(\sum_{k=0}^{m} \left\| \frac{\mathrm{d}^k}{\mathrm{d}x^k} v(x) \right\|_{0,D}^2\right)^{1/2}$$

is bounded, where as usual, $\frac{\mathrm{d}}{\mathrm{d}x}$ signifies the derivative operator in the weak sense. By $H_0^1(D)$ we denote the subspace of $H^1(D)$ consisting of all functions with zero boundary trace. Moreover, let

$$\|v\|_{\infty,D} = \operatorname*{ess\,sup}_{x\in D} |v(x)|,$$

with associated space $L^\infty(D) = \{v : D \to \mathbb{R} : \|v\|_{\infty,D} < \infty\}$.

The basis of the new *hp*-adaptive strategy to be presented in this paper is the following embedding result.

Proposition 1. *Let $(a, b) \subset \mathbb{R}$, $a < b$, be an interval, and $h = b - a$. Then, the embedding $H^1(a, b) \hookrightarrow L^\infty(a, b)$ is continuous, more precisely, for any $u \in H^1(a, b)$ there holds the inequality*

$$\|u\|_{\infty,(a,b)}^2 \le \coth(1) \left(h^{-1} \|u\|_{0,(a,b)}^2 + h \left\|u'\right\|_{0,(a,b)}^2\right). \tag{1}$$

Here, $\coth(.)$ *is the hyperbolic cotangent function.*

Proof. According to [13, Theorem 1], it holds that

$$\|u\|_{\infty,(0,1)}^2 \le \coth(1)\|u\|_{1,(0,1)}^2;$$

see also [14,15]. Then, using a standard scaling argument, we obtain the bound (1). □

Remark 2. Note that the estimate (1) is sharp. Indeed, there holds equality for the function $u(x) = \cosh\left(\frac{x-a}{h}\right)$, where $\cosh(\cdot)$ is the hyperbolic cosine.

3. *hp*-adaptive finite element methods

In the following elaborations, we shall restrict ourselves to one space dimension. We note, however, that our *hp*-adaptive strategy can be generalized immediately to higher dimensions; see Remark 4 below.

3.1. hp-discretization and model problem

Let $\Omega = (0, 1) \subset \mathbb{R}$. Then, for an integer $N \in \mathbb{N}$, let us consider a subdivision (mesh) $\mathcal{T}_N = \{K_j\}_{j=1}^N$ of Ω into N open subintervals (elements) $K_j = (x_{j-1}, x_j)$, $j = 1, 2, \ldots, N$. Here, $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$. The size of an element is denoted by $h_j = |K_j| = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$. Furthermore, to each element K_j, $j = 1, 2, \ldots, N$, we associate a polynomial degree $p_j \ge 1$. These quantities are stored in a polynomial degree vector $\boldsymbol{p}_N = (p_1, p_2, \ldots, p_N)$.

Based on a finite element mesh $\mathcal{T}_N$ with associated polynomial degree vector $\boldsymbol{p}_N$, we define the *hp*-space

$$V(\mathcal{T}_N, \boldsymbol{p}_N) = \{v \in H_0^1(\Omega) : v|_{K_j} \in \mathbb{P}_{p_j}(K_j), j = 1, 2, \dots, N\},$$

where $\mathbb{P}_{p_j}(K_j)$ denotes the space of all polynomials of degree at most p_j on each element K_j, $j = 1, 2, \dots, N$.

Let us consider the simple 1D elliptic boundary value model problem

$$-au'' + \mathrm{d}u = f \quad \text{on } \Omega, \qquad u(0) = u(1) = 0, \tag{2}$$

with a right-hand side $f \in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, and two constants $a > 0$, $d \geq 0$. We define a corresponding energy norm by

$$\|v\|_{E,(0,1)}^2 = a\left\|v'\right\|_{0,(0,1)}^2 + d\|v\|_{0,(0,1)}^2, \quad v \in H_0^1(0,1).$$

The above problem has a unique solution in $H_0^1(\Omega)$ which we shall approximate by the following standard *hp*-finite element formulation (cf. [9,2]): Find a function $u_{\mathrm{hp}} \in V(\mathcal{T}_N, \boldsymbol{p}_N)$ such that

$$a\int_0^1 u_{\mathrm{hp}}' v' \,\mathrm{d}x + d\int_0^1 u_{\mathrm{hp}} v \,\mathrm{d}x = \int_0^1 f v \,\mathrm{d}x \tag{3}$$

for any $v \in V(\mathcal{T}_N, \boldsymbol{p}_N)$.

3.2. hp a posteriori error estimates

We briefly recall an a posteriori error estimation result from, e.g., [2, Section 3.5.2] and [3].

Proposition 3. *Let $u \in H_0^1(0,1)$ signify the solution of (2), and $u_{\mathrm{hp}} \in V(\mathcal{T}_N, \boldsymbol{p}_N)$ the hp-FEM solution from (3) on any mesh $\mathcal{T}_N = \{K_j\}_{j=1}^N$ and any polynomial degree vector $\boldsymbol{p}_N = (p_1, p_2, \dots, p_N)$. Then, there holds the following upper a posteriori error bound:*

$$\left\|u - u_{\mathrm{hp}}\right\|_{E,(0,1)}^2 \leq \sum_{j=1}^N \eta_{K_j}^2 + \frac{1}{ap_j(p_j+1)}\left\|(f - \Pi_N f)\omega_j^{1/2}\right\|_{0,K_j}^2. \tag{4}$$

Here,

$$\eta_{K_j} = \frac{1}{\sqrt{ap_j(p_j+1)}}\left\|(\Pi_N f + au_{\mathrm{hp}}'' - du_{\mathrm{hp}})\omega_j^{1/2}\right\|_{0,K_j}$$

is the local error estimator on K_j, $j = 1, 2, \dots, N$. Moreover, Π_N denotes the L^2-projection onto $V(\mathcal{T}_N, \boldsymbol{p}_N)$, and ω_j is a non-negative weight function on $K_j = (x_{j-1}, x_j)$ given by $\omega_j(x) = (x_j - x)(x - x_{j-1})$.

In addition, the local lower bound

$$\eta_{K_j} \leq c\left(a\left\|u' - u_{\mathrm{hp}}'\right\|_{0,K_j}^2 + d\left\|u - u_{\mathrm{hp}}\right\|_{0,K_j}^2\right)^{1/2} + \frac{1}{\sqrt{ap_j(p_j+1)}}\left\|(f - \Pi_N f)\omega_j^{1/2}\right\|_{0,K_j},$$

for $j = 1, 2, \dots, N$, holds true, with a constant $c > 0$ independent of $\mathcal{T}_N$ and $\boldsymbol{p}_N$.

The above a posteriori error estimates show that the local error estimators η_{K_j} are both reliable and efficient. Consequently, proceeding in a standard way (cf., e.g, [16]) they can be applied to mark all elements in the finite element mesh $\mathcal{T}_N$ for which comparatively large error contributions occur. The main issue in *hp*-adaptivity is to decide, for each marked element, whether *h*- or *p*-refinement is more suitable, i.e., whether element bisection or increasing the local polynomial degree is more beneficial in the numerical approximation. In the following section, we shall present a new approach in order to address this question.

3.3. hp-adaptive strategy

Let us consider an open subset $K \subseteq \Omega$ with size $h_K = |K|$ and a function $u \in H^1(K)$. Then, in accordance with (1), we define the quantity

$$\mathcal{F}_K[u] := \begin{cases} \|u\|_{\infty,K}^2\left[\coth(1)\left(h_K^{-1}\|u\|_{0,K}^2 + h_K\left\|u'\right\|_{0,K}^2\right)\right]^{-1} & \text{if } u \not\equiv 0, \\ 1 & \text{if } u \equiv 0. \end{cases}$$

Recalling Proposition 1, there holds

$$0 \leq \mathcal{F}_K[u] \leq 1. \tag{5}$$

We now ask the question whether the value of $\mathcal{F}_K[u]$ is connected with the smoothness of u. Intuitively, if u is smooth on K, we would expect that $\left\|u'\right\|_{0,K}$ is comparatively small. In that case, the value of $\mathcal{F}_K$ is relatively close to 1. In contrast, if u varies strongly on K or has a steep derivative, we would suppose that $\mathcal{F}_K[u]$ is much closer to 0.

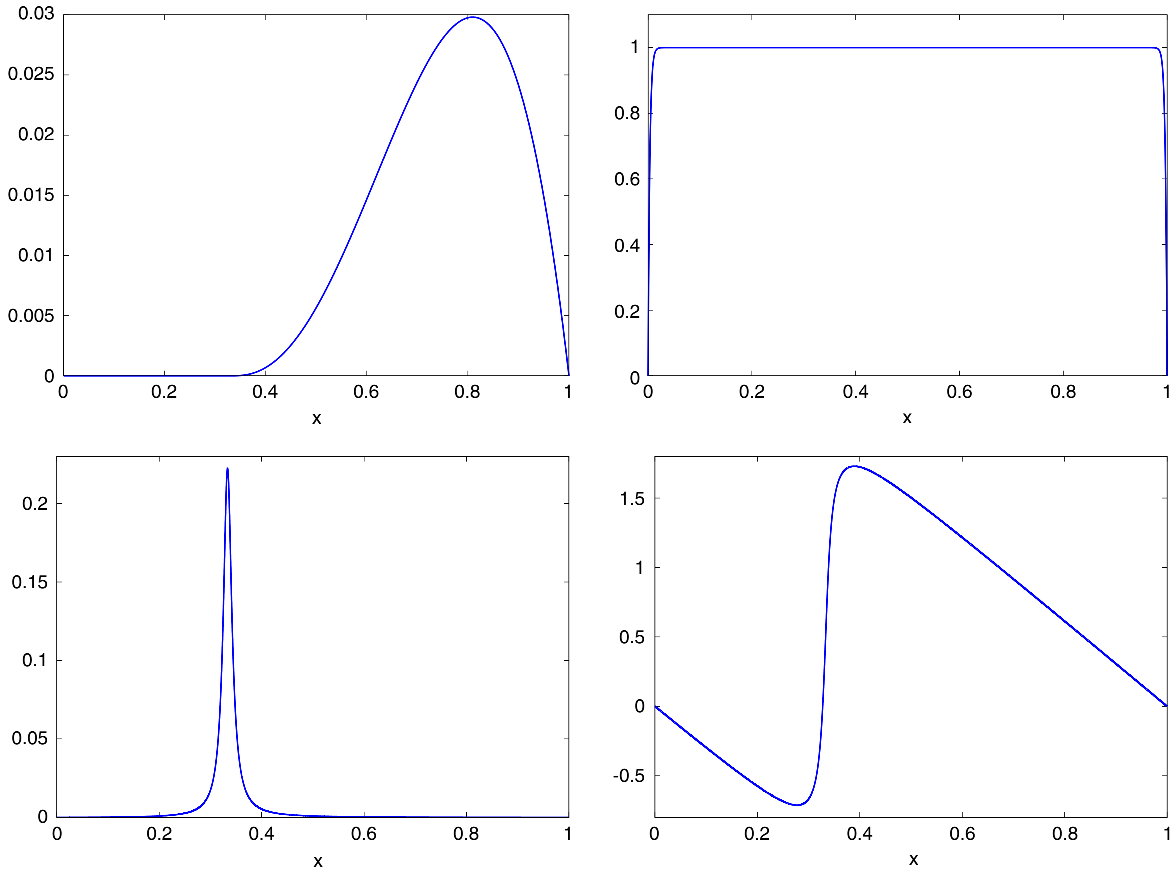


Fig. 1. Exact solutions for Examples 1–4.

Let us illustrate this observation by means of some simple examples on the unit interval $\widehat{K} = (0, 1)$. For any integer $n \in \mathbb{N}$, we consider the functions

$$u_n(x) = \cos(\pi n x), \qquad v_n(x) = x^n,$$
$$w_n(x) = \left(1 + 10^n (x - 1/2)^2\right)^{-1}, \qquad z_n(x) = \arctan\left(10^n (x - 1/2)\right).$$

Note that, as $n \to \infty$, u_n features an increasing number of oscillations, v_n develops a steep derivative, w_n exhibits a spike at $x = 0.5$, and z_n forms a shock at $x = 0.5$. Computing the corresponding values of $\mathcal{F}$ on $(0, 1)$, we find:

n	1	2	3	4	5
$\mathcal{F}_{(0,1)}[u_n]$	0.1401	0.0376	0.0170	0.0096	0.0061
$\mathcal{F}_{(0,1)}[v_n]$	0.5712	0.4967	0.3920	0.3177	0.2655
$\mathcal{F}_{(0,1)}[w_n]$	0.2771	0.0952	0.0306	0.0097	0.0031
$\mathcal{F}_{(0,1)}[z_n]$	0.0852	0.0115	0.0012	0.0001	0.00001

Evidently, the greater the n is, i.e., the more the derivatives change locally, the smaller the value of $\mathcal{F}_{(0,1)}$ becomes. This confirms the previously mentioned intuition that as $\mathcal{F}_K[u]$ increases from 0 to 1 the better the u behaves.

The same idea applies to the derivatives of u. In particular, for smooth functions we would expect that $\mathcal{F}_K[u^{(k)}]$ is clearly bounded away from 0 for some range of derivative orders $k = 1, 2, \ldots, k_{\max}$. Consequently, a function $u \in H^m(K)$, for some $m \in \mathbb{N}$, with comparatively high regularity would be expected to satisfy

$$0 \ll \mathcal{F}_K[u^{(k)}] \leq 1, \quad k = 0, 1, 2, \ldots, m - 1. \tag{6}$$

Conversely, from the point of view of approximating u on a subdomain K by suitable hp-approximations, the value of $\mathcal{F}$ could be used in order to obtain some indication on the smoothness of u. Specifically, if $\mathcal{F}_K[u] \leq 1$ is sufficiently large, then u may be considered smooth on K, and it is reasonable to increase the polynomial degree in a piecewise polynomial approximation. Otherwise, if the value of $\mathcal{F}_K[u]$ is small, then the element K is bisected into two new elements in order to improve the approximation of u by a piecewise polynomial.

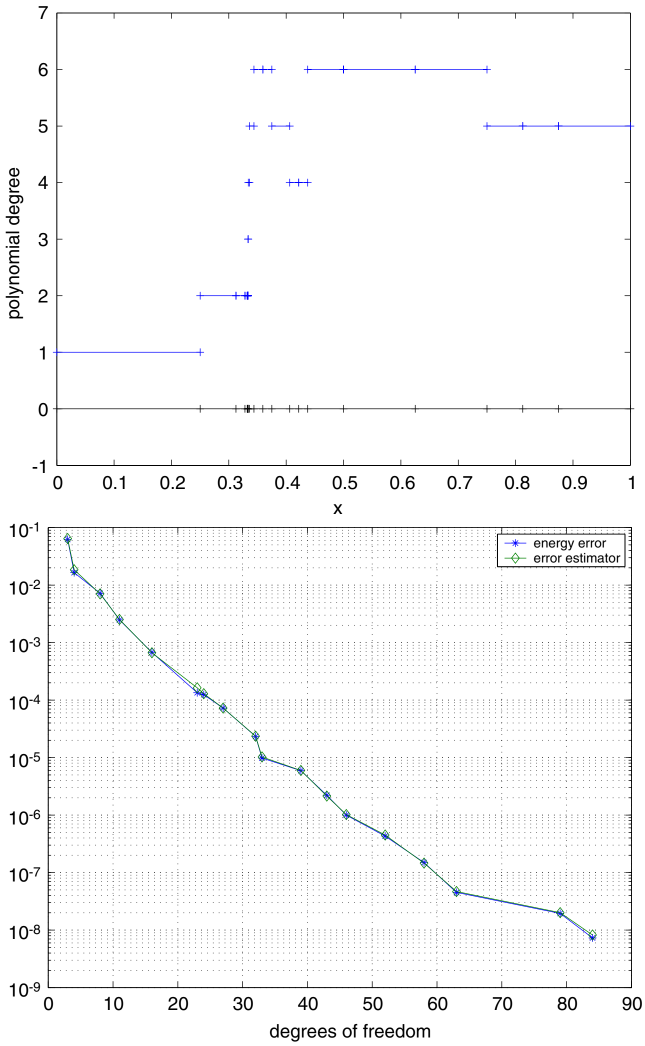


Fig. 2. Example 1: *hp*-mesh after 18 refinement steps (top) and performance of *hp*-FEM (bottom).

We adopt this idea to the *hp*-finite element context. Naturally, any finite element function $u_{\mathrm{hp}} \in V(\mathcal{T}_N, \boldsymbol{p}_N)$, where $\mathcal{T}_N$ is a finite element partition of a domain Ω and $\boldsymbol{p}_N$ is an associated polynomial degree vector, is a piecewise polynomial and hence locally smooth. Having in mind, however, that u_{hp} is the approximation of the exact solution u of (2), we might get an indication on the smoothness of u by testing inequality (6) for u_{hp}. To this end, for a fixed parameter $\tau \in [0, 1]$, any element $K \in \mathcal{T}_N$ and any integer $k \in \mathbb{N}_0$, let us define the set

$$\widetilde{H}^k_\tau(K) := \left\{ v \in H^k(K) : \mathcal{F}_K[v^{(k-1)}] \geq \tau \right\}. \tag{7}$$

Using this definition, we might say that u_{hp} is classified as smooth if $u_{\mathrm{hp}} \in \widetilde{H}^k_\tau(K)$, and nonsmooth otherwise. This idea motivates a simple decision process for elementwise *h*- and *p*-refinement. More precisely, given the finite element solution u_{hp} from (3) on an *hp*-FEM space $V(\mathcal{T}_N, \boldsymbol{p}_N)$ the following algorithm performs a single *hp*-refinement step.

Algorithm 1 (*hp-refinement*)**.** For all elements $K_j \in \mathcal{T}_N$ do

- if K_j is marked for refinement then
  - if $u_{\mathrm{hp}}|_{K_j} \in \widetilde{H}^{p_j}_\tau(K_j)$ then the polynomial degree on K_j is increased by 1, i.e., $p_j \leftarrow p_j + 1$.
  - else bisect K_j into two new elements and assign the polynomial degree p_j to each of these new elements.
- else leave K_j and p_j unchanged.

end do.

The output of this scheme is a refined *hp*-space that can be enriched further in the same way.

Combining the above *hp*-refinement procedure with a standard marking strategy based on the local error estimators η_{K_j} from Proposition 3, results in the ensuing *hp*-adaptive finite element method. Here, $\theta \in (0, 1)$ is a given marking parameter.

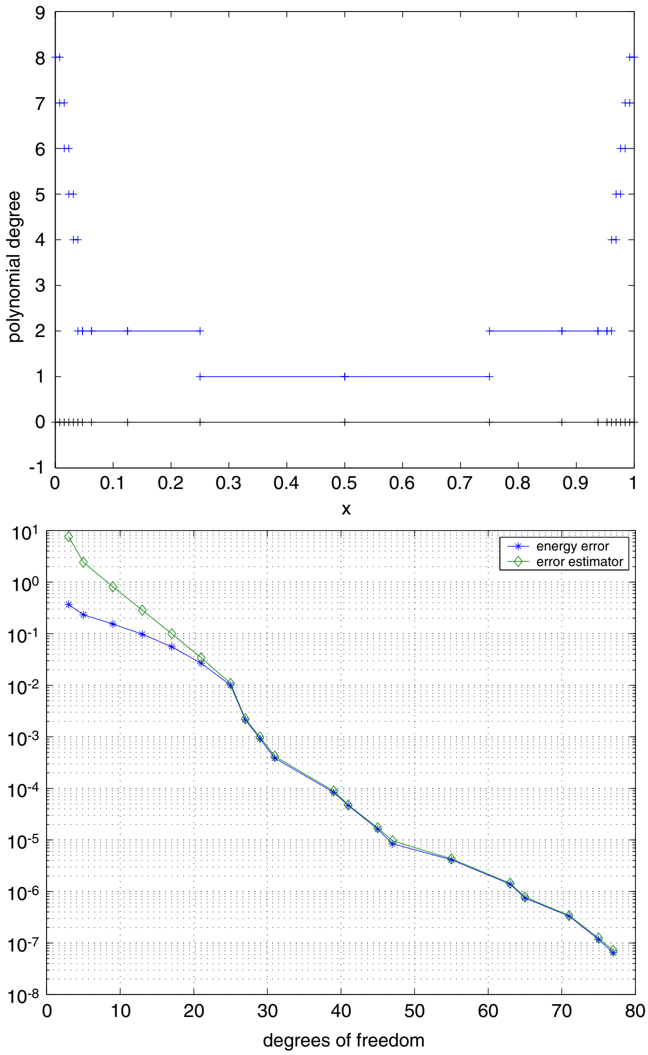


Fig. 3. Example 2: *hp*-mesh after 20 refinement steps (top) and performance of *hp*-FEM (bottom).

Algorithm 2 (*hp-adaptive FEM*).

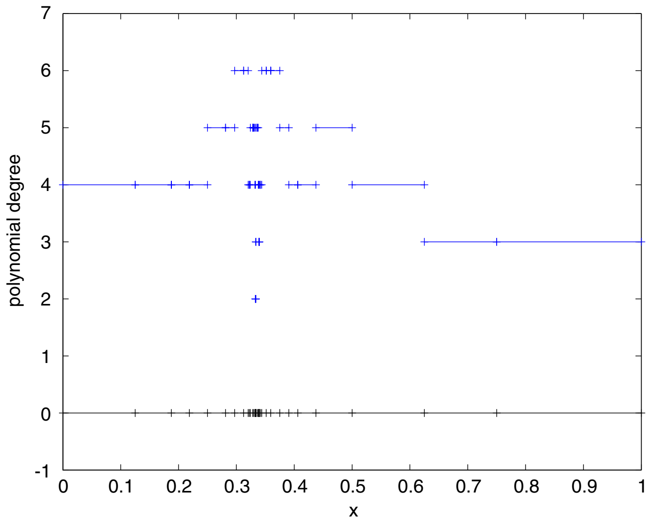
0. Set $N = 1$. Choose an initial coarse mesh $\mathcal{T}_1$ and an associated polynomial degree vector $\boldsymbol{p}_1$.
1. Compute the FEM solution $u_{\mathrm{hp}}^{(N)}$ on the *hp*-space $V(\mathcal{T}_N, \boldsymbol{p}_N)$.
2. Mark all elements $\widetilde{K} \in \mathcal{T}_N$ for which there holds

$$\max_{K \in \mathcal{T}_N} \eta_K \le \theta^{-1} \eta_{\widetilde{K}}.$$

3. Apply Algorithm 1 to $u_{\mathrm{hp}}^{(N)}$ on all marked elements in order to obtain a refined mesh $\mathcal{T}_{N+1}$ and a new associated polynomial degree vector $\boldsymbol{p}_{N+1}$.
4. Set $N \leftarrow N + 1$, and go to step 1. Iterate until the global error (may be estimated using (4)) is smaller than a prescribed tolerance $\mathrm{tol} > 0$.

Remark 4. The proposed *hp*-adaptive procedure offers a number of advantages:

1. In the current work we suggest the use of the Sobolev inequality (1). More generally, other continuous Sobolev embeddings, such as, e.g., $H^1 \hookrightarrow L^q$ could be applied. This is particularly interesting in higher dimensions where H^1-functions are not necessarily bounded. In this context, it should be emphasized that the proposed idea for smoothness testing is not restricted to one space dimension. In fact, it can be used easily in higher dimensions and may be applied to elements of various shapes (based on appropriate continuous Sobolev embeddings; see, e.g., [17]).



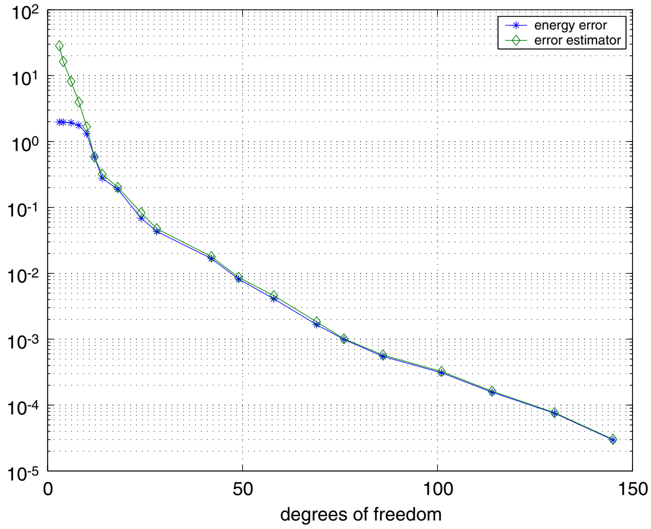


Fig. 4. Example 3: *hp*-mesh after 20 refinement steps (top) and performance of *hp*-FEM (bottom).

2. Furthermore, we note that the approach in this paper does not depend on the underlying PDE problem and its variational formulation. In addition, it can be combined with other finite element methods (such as non-conforming or mixed methods) as long as appropriate error indicators are available.
3. Finally, we remark that smoothness testing in this work is performed element-by-element and consequently, it is computationally inexpensive.

4. Numerical experiments

The goal of this section is to test Algorithm 2 by means of four examples on the domain $\Omega = (0, 1)$. In all computations, the initial mesh consists of 4 elements of equal size $h = 1/4$, and the polynomial degree is chosen to be 1 on each element. Furthermore, we let the marking parameter in Algorithm 2 to be $\theta = 0.5$ and the smoothness parameter in (7) is set to be $\tau = 0.5$. The exact solutions of the problems are shown in Fig. 1.

Example 1. We first consider $a = 1$, $d = 0$ in (2) and choose the source term f such that the exact solution $u \in H_0^1(0, 1)$ becomes

$$u(x) = \begin{cases} 0 & \text{for } x \in (0, 1/3] \\ (x - 1/3)^{5/2}(1 - x) & \text{for } x \in (1/3, 1). \end{cases}$$

Note that the solution is smooth except for a local singularity at $x = 1/3$. In Fig. 2 we present the *hp*-mesh after 18 refinement steps and the behavior of the energy errors and error estimators.

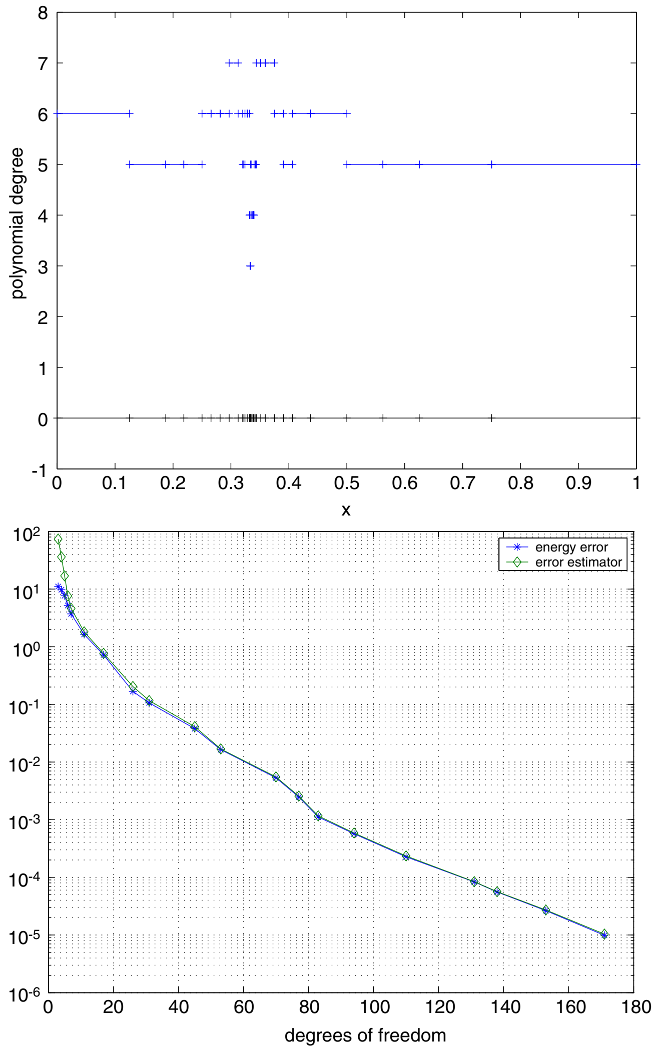


Fig. 5. Example 4: *hp*-mesh after 20 refinement steps (top) and performance of *hp*-FEM (bottom).

Example 2. Here, we choose $f = 1$ and $d = 1$ in (2). Then, the exact solution is given by

$$u(x) = -\frac{\exp(x/\sqrt{a})}{\exp(1/\sqrt{a}) + 1} - \frac{\exp(-x/\sqrt{a})\exp(1/\sqrt{a})}{\exp(1/\sqrt{a}) + 1} + 1.$$

If $0 < a \ll 1$, then u exhibits a boundary layer with steep derivative close to $x = 0$ and $x = 1$. In our experiment, we choose $a = 10^{-5}$. The corresponding results are displayed in Fig. 3 for 20 refinement steps.

Example 3. Here, the exact solution,

$$u(x) = \frac{x(1-x)}{1 + 10^4(x - 1/3)^2},$$

exhibits a spike at $x = 1/3$. We let $a = d = 1$, and choose f in (2) accordingly. The resulting computations are depicted in Fig. 4.

Example 4. In the last experiment, for $a = d = 1$, we consider the shock solution

$$u(x) = \arctan(s(x - 1/3)) + (1 - x)\arctan(s/3) - x\arctan(2s/3),$$

with $s = 100$. In Fig. 5 the computational results have been plotted.

Discussion of results. We clearly see that the *hp*-adaptive algorithm as applied to the examples above, is capable of locating and properly resolving the local irregularities arising in the exact solutions. Indeed, the meshes are strongly refined close to those points, and exponential convergence is obtained in all test cases. We also observe that in areas where the exact solution is flat, large elements and low polynomial degrees suffice for a good approximation.

Furthermore, we remark that more or less *p*-refinement could be kindled in Algorithm 1 by varying the value of the smoothness parameter τ. In addition, slightly steeper slopes in the exponential decay of the energy errors can be obtained by appropriately adjusting τ to the underlying problem. The emphasis in the experiments, however, was to illustrate that the algorithm performs well even without changing the parameters involved.

5. Conclusions

In this article, we have introduced a new *hp*-adaptive refinement strategy for FEM based on smoothness testing by means of Sobolev inequalities. The procedure is easily extensible to higher dimensions and arbitrarily shaped elements (with, for example, Lipschitz continuous boundary). Indeed, due to its local character, the algorithm will be particularly interesting in two and three dimensions. This will be the subject of a forthcoming work. In addition, we note that the *hp*-decision process presented here does neither depend on the underlying PDE problem nor on the finite element discretization (as long as suitable error estimators are available). Numerical experiments in one space dimension confirm that the proposed scheme is able to locate local singularities and to generate sequences of *hp*-spaces which result in exponentially convergent finite element solutions.

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